Soit $n \in \mathbb{N}$, supposons $V_n = \left(\frac{1}{3}\right)^{(2^n)}$ Montrons que : $V_{n+1} = \left(\frac{1}{3}\right)^{(2^{n+1})}$

On a : $V_{n+1} = V_n^2 = \left[\left(\frac{1}{3}\right)^{(2^n)}\right]^2 = \left(\frac{1}{3}\right)^{(2^n)\times 2} = \left(\frac{1}{3}\right)^{(2^{n+1})}$

Conclusion : Pour $\forall n \in \mathbb{N}$: $V_n = \left(\frac{1}{3}\right)^{(2^n)}$

4°/

a) On a : $U_{n+1} < 1$ d'où : $1 - U_{n+1} > 0$

$$1 - U_{n+1} = 1 - \frac{2U_n}{1+U_n^2} = \frac{(1-U_n)^2}{1+U_n^2}$$

$$(1 - U_{n+1}) - \frac{4}{5}(1 - U_n) = \frac{(1-U_n)^2}{1+U_n^2} - \frac{4}{5}(1 - U_n)$$

$$= (1 - U_n)\left[\frac{1-U_n}{1+U_n^2} - \frac{4}{5}\right] = (1 - U_n)\left(\frac{5-5U_n-4-4U_n^2}{5(1+U_n)}\right)$$

$$= \frac{(1-U_n)(1-5U_n-4U_n^2)}{5(1+U_n)}$$

Soit $P(x) = -4x^2 - 5x + 1$

$\Delta = 25 + 16 = 41$

$x' = \frac{5-\sqrt{41}}{-8}$; $x'' = \frac{5+\sqrt{41}}{-8}$

x		x''		x'	
P(x)	-	0	+	0	-

Or $U_n \in \left[\frac{1}{2}, 1\right]$ d'où : $1 - 5U_n - 4U_n^2 < 0$

($x' < \frac{1}{2} < 1$) et par suite $0 < 1 - U_{n+1} < \frac{4}{5}(1 - U_n)$